

Difference Krichever–Novikov operators

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Abstract

In this paper we study commuting difference operators of rank two. We introduce an equation on potentials $V(n), W(n)$ of the difference operator $L_4 = (T + V(n)T^{-1})^2 + W(n)$ and some additional data. With the help of this equation we find the first examples of commuting difference operators of rank two corresponding to spectral curves of higher genus.

1 Introduction and main results

I.M. Krichever and S.P. Novikov [1], [2] discovered a remarkable class of solutions of soliton equations — algebro–geometric solutions of rank $l > 1$. This class is determined by the following condition: common eigenfunctions of auxiliary commuting ordinary differential or difference operators form a vector bundle of rank l over the spectral curve. Rank two solutions of the Kadomtsev–Petviashvili (KP) equation and 2D-Toda chain corresponding to the spectral curves of genus $g = 1$ were found in [1], [2]. To find higher rank solutions one has to find higher rank commuting operators and their appropriate deformations. The problem of classification of commuting differential and difference operators was solved in [2]–[4]; however, finding the operators themselves has remained an open problem. Moreover, no examples of commuting differential operators of rank $l > 1$ at $g > 1$ were known before the recent paper [5] (see also [6]).

In this paper we study commuting difference operators. We denote by L_k, L_s the operators of orders $k = N_- + N_+$ and $s = M_- + M_+$

$$L_k = \sum_{j=N_-}^{N_+} u_j(n)T^j, \quad L_s = \sum_{j=M_-}^{M_+} v_j(n)T^j, \quad n \in \mathbb{Z},$$

where T is the shift operator. The condition of their commutativity is equivalent to a complicated system of nonlinear difference equations on the coefficients. These equations have been studied since the beginning of the 20th century (see [7]). An analogue of the Burchnell–Chaundy lemma [8] holds. Namely, if $L_k L_s = L_s L_k$, then there exists a nonzero polynomial $F(z, w)$ such that $F(L_k, L_s) = 0$ [9]. The polynomial F defines the *spectral curve*

$$\Gamma = \{(z, w) \in \mathbb{C}^2 | F(z, w) = 0\}.$$

The spectral curve parametrizes common eigenvalues, i.e. if

$$L_k\psi = z\psi, \quad L_s\psi = w\psi,$$

then $(z, w) \in \Gamma$. The dimension of the space of common eigenfunctions with the fixed eigenvalues is called the *rank* of the pair L_k, L_s

$$l = \dim\{\psi : L_k\psi = z\psi, \quad L_s\psi = w\psi\},$$

where the point $(z, w) \in \Gamma$ is in general position. Thus the spectral curve and the rank are defined exactly the same way as in the case of the differential operators.

Before discussing difference operators we briefly discuss differential operators. The first important results relating to commuting differential operators of rank $l > 1$ were obtained by J. Dixmier [10] and V.G. Drinfel'd [11]. The operators of rank l with periodic coefficients were studied in [12]. Rank two operators at $g = 1$ were found by I.M. Krichever and S.P. Novikov [1]. These operators were studied in [13]–[20] (see also [21]–[24] for $g = 2-4, l = 2, 3$). At $g = 1, l = 3$ operators were found in [25]. Methods of [5] allow to construct and study higher rank operators at $g > 1$ [26]–[31].

The maximal commutative ring of difference operators containing L_k and L_s is isomorphic to the ring of meromorphic functions on an algebraic spectral curve Γ with poles $q_1, \dots, q_m \in \Gamma$ (see [2]). Such operators are called *m-points operators*. We note that any ring of commuting differential operators is isomorphic to a ring of meromorphic functions on a spectral curve with a unique pole. The commuting difference operators of rank one were found by I.M. Krichever [9] and D. Mumford [32]. Eigenfunctions (Baker–Akhiezer functions) and coefficients of such operators can be found explicitly with the help of theta-functions of the Jacobi varieties of spectral curves. In the case of $l > 1$ eigenfunctions cannot be found explicitly. Finding such operators is still an open problem. Rank two one-point operators at $g = 1$ were found in [2], operators with polynomial coefficients among them were obtained in [33].

In this paper we consider one-point operators of rank two L_4, L_{4g+2} corresponding to the hyperelliptic spectral curve Γ

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + c_{2g-1}z^{2g-1} + \dots + c_0, \quad (1)$$

herewith

$$L_4 = \sum_{i=-2}^2 u_i(n)T^i, \quad L_{4g+2} = \sum_{i=-(2g+1)}^{2g+1} v_i(n)T^i, \quad u_2 = v_{2g+1} = 1, \quad (2)$$

$$L_4\psi = z\psi, \quad L_{4g+2}\psi = w\psi, \quad \psi = \psi(n, P), \quad P = (z, w) \in \Gamma. \quad (3)$$

Common eigenfunctions of L_4 and L_{4g+2} satisfy the equation

$$\psi(n+1, P) = \chi_1(n, P)\psi(n-1, P) + \chi_2(n, P)\psi(n, P), \quad (4)$$

where $\chi_1(n, P)$ and $\chi_2(n, P)$ are rational functions on Γ having $2g$ simple poles, depending on n (see [2]). The function $\chi_2(n, P)$ additionally has a simple pole at $q = \infty$. To

find L_4 and L_{4g+2} it is sufficient to find χ_1 and χ_2 . Let σ be the holomorphic involution on Γ , $\sigma(z, w) = \sigma(z, -w)$. The main results of this paper are Theorems 1–4.

Theorem 1 *If*

$$\chi_1(n, P) = \chi_1(n, \sigma(P)), \quad \chi_2(n, P) = -\chi_2(n, \sigma(P)), \quad (5)$$

then L_4 has the form

$$L_4 = (T + V_n T^{-1})^2 + W_n, \quad (6)$$

where

$$\chi_1 = -V_n \frac{Q_{n+1}}{Q_n}, \quad \chi_2 = \frac{w}{Q_n}, \quad Q_n(z) = z^g + \alpha_{g-1}(n)z^{g-1} + \dots + \alpha_0(n). \quad (7)$$

Functions V_n, W_n, Q_n satisfy

$$F_g(z) = Q_{n-1}Q_{n+1}V_n + Q_nQ_{n+2}V_{n+1} + Q_nQ_{n+1}(z - V_n - V_{n+1} - W_n). \quad (8)$$

In Theorem 1 and further we use the notations $V_n = V(n), W_n = W(n)$. It is a remarkable fact that (8) can be linearized. Namely, if we replace $n \rightarrow n+1$ and take the difference with (8), then the result can be divided by $Q_{n+1}(z)$. Finally we obtain the linear equation on $Q_n(z)$.

Corollary 1 *Functions $Q_n(z), V_n, W_n$ satisfy*

$$Q_{n-1}V_n + Q_n(z - V_n - V_{n+1} - W_n) - Q_{n+2}(z - V_{n+1} - V_{n+2} - W_{n+1}) - Q_{n+3}V_{n+2} = 0. \quad (9)$$

At $g = 1$, the equation (8) allows us to express V_n, W_n via a functional parameter γ_n .

Corollary 2 *The operator $L_4 = (T + V_n T^{-1})^2 + W_n$, where*

$$V_n = \frac{F_1(\gamma_n)}{(\gamma_n - \gamma_{n-1})(\gamma_n - \gamma_{n+1})}, \quad W_n = -c_2 - \gamma_n - \gamma_{n+1}, \quad (10)$$

commutes with

$$L_6 = T^3 + (V_n + V_{n+1} + V_{n+2} + W_n - \gamma_{n+2})T + \\ + V_n(V_{n-1} + V_n + V_{n+1} + W_n - \gamma_{n-1})T^{-1} + V_{n-2}V_{n-1}V_n T^{-3}.$$

The spectral curve of L_4, L_6 is $w^2 = F_1(z)$.

In the theory of commuting ordinary differential operators there are equations which are similar to (8), (9). Let us compare (8), (9) with their smooth analogues. First, we consider the one-dimensional finite-gap Schrödinger operator $\mathcal{L}_2 = -\partial_x^2 + \mathcal{V}(x)$ commuting with a differential operator \mathcal{L}_{2g+1} of order $2g+1$. The theory of such operators is

closely related to the theory of periodic and quasiperiodic solutions of the Korteweg–de Vries equation (see [34]–[36]). Denote by ψ a common eigenfunction

$$(-\partial_x^2 + \mathcal{V}(x))\psi = z\psi, \quad \mathcal{L}_{2g+1}\psi = w\psi.$$

The point $P = (z, w)$ belongs to the spectral curve (1). Function $\psi(x, P)$ satisfies

$$\psi'(x, P) = i\chi_0(x, P)\psi(x, P),$$

where

$$\chi_0 = \frac{\mathcal{Q}_x}{2i\mathcal{Q}} + \frac{w}{\mathcal{Q}}, \quad \mathcal{Q} = z^g + \alpha_{g-1}(x)z^{g-1} + \dots + \alpha_0(x).$$

Polynomial \mathcal{Q} satisfies the equation

$$4F_g(z) = 4(z - \mathcal{V})\mathcal{Q}^2 - (\mathcal{Q}_x)^2 + 2\mathcal{Q}\mathcal{Q}_{xx},$$

which is linearized as well as (8) (see [37], [38])

$$\mathcal{Q}_{xxx} - 4\mathcal{Q}_x(\mathcal{V} - z) - 2\mathcal{V}_x\mathcal{Q} = 0.$$

Equations (8), (9) are analogues of the last two.

Let us consider one more example. We denote by \mathcal{L}_4 , \mathcal{L}_{4g+2} rank two commuting differential operators with the spectral curve (1). The common eigenfunctions of \mathcal{L}_4 and \mathcal{L}_{4g+2} satisfy

$$\psi'' = \chi_1(x, P)\psi' + \chi_0(x, P)\psi.$$

In [5] it was proved that \mathcal{L}_4 is self-adjoint if and only if $\chi_1(x, P) = \chi_1(x, \sigma(P))$, herewith

$$\mathcal{L}_4 = (\partial_x^2 + \mathcal{V}(x))^2 + \mathcal{W}(x),$$

$$\chi_0 = -\frac{1}{2}\frac{\mathcal{Q}_{xx}}{\mathcal{Q}} + \frac{w}{\mathcal{Q}} - \mathcal{V}, \quad \chi_1 = \frac{\mathcal{Q}_x}{\mathcal{Q}}, \quad \mathcal{Q} = z^g + \alpha_{g-1}(x)z^{g-1} + \dots + \alpha_0(x),$$

polynomial \mathcal{Q} satisfies

$$4F_g(z) = 4(z - \mathcal{W})\mathcal{Q}^2 - 4\mathcal{V}(\mathcal{Q}_x)^2 + \mathcal{Q}_{xx}^2 - 2\mathcal{Q}_x\mathcal{Q}_{xxx} + 2\mathcal{Q}(2\mathcal{V}_x\mathcal{Q}_x + 4\mathcal{V}\mathcal{Q}_{xx} + \mathcal{Q}_{xxxx}), \quad (11)$$

and also satisfies

$$\partial_x^5 \mathcal{Q} + 4\mathcal{V}\mathcal{Q}_{xxx} + 2\mathcal{Q}_x(2z - 2\mathcal{W} - \mathcal{V}_{xx}) + 6\mathcal{V}_x\mathcal{Q}_{xx} - 2\mathcal{Q}\mathcal{W}_x = 0. \quad (12)$$

Equations (8), (9) are discrete analogues of (11), (12).

Theorem 1 allows us to construct the examples.

Theorem 2 *The operator*

$$L_4^\sharp = (T + (r_3n^3 + r_2n^2 + r_1n + r_0)T^{-1})^2 + g(g+1)r_3n, \quad r_3 \neq 0$$

commutes with a difference operator L_{4g+2}^\sharp .

Theorem 3 *The operator*

$$L_4^\vee = (T + (r_1 a^n + r_0)T^{-1})^2 + r_1(a^{2g+1} - a^{g+1} - a^g + 1)a^{n-g}, \quad r_1, a \neq 0,$$

where $a^{2g+1} - a^{g+1} - a^g + 1 \neq 0$, commutes with a difference operator L_{4g+2}^\vee .

Theorem 4 *The operator*

$$L_4^\natural = (T + (r_1 \cos(n) + r_0)T^{-1})^2 - 4r_1 \sin\left(\frac{g}{2}\right) \sin\left(\frac{g+1}{2}\right) \cos\left(n + \frac{1}{2}\right), \quad r_1 \neq 0$$

commutes with a difference operator L_{4g+2}^\natural .

In Section 2 we recall the Krichever–Novikov equations on Tyurin parameters.

In Section 3 we prove Theorems 1–4 and consider examples.

In Appendix we consider the differential–difference system on $V_n(t)$, $W_n(t)$

$$\dot{V}_n = V_n(W_{n-1} - W_n + V_{n-1} - V_{n+1}), \quad (13)$$

$$\dot{W}_n = (W_n - W_{n-1})V_n + (W_{n+1} - W_n)V_{n+1}. \quad (14)$$

From (13), (14) it follows that $\varphi_n(t)$, where $e^{\varphi_n(t)} = V_n(t)$, satisfies the generalized Toda chain

$$\ddot{\varphi}_n = e^{\varphi_{n-2} + \varphi_{n-1}} - e^{\varphi_{n-1} + \varphi_n} + e^{\varphi_{n+1} + \varphi_{n+2}} - e^{\varphi_{n+1} + \varphi_n}.$$

From (13), (14) it follows also that

$$[L_4, \partial_t - V_{n-1}(t)V_n(t)T^{-2}] = 0,$$

where $L_4 = (T + V_n(t)T^{-1})^2 + W_n(t)$. Following [1], [2] we call the solution $V_n(t), W_n(t)$ of (13), (14) *the solution of rank two*, if additionally $[L_4, L_{4g+2}] = 0$ for some difference operator L_{4g+2} . In the case of rank two solutions an evolution equation on $Q_n(t)$ is obtained in Theorem 5. At $g = 1$ this equation is reduced to a discrete analogue of the Krichever–Novikov equation, which appeared in the theory of rank two solutions of KP.

2 Discrete dynamics of the Tyurin parameters

As mentioned above, in the case of rank one operators the eigenfunctions can be found explicitly in terms of theta-functions of the Jacobi varieties of spectral curves. Let us consider the simplest example. Let Γ be an elliptic curve $\Gamma = \mathbb{C}/\{\mathbb{Z} + \tau\mathbb{Z}\}$, $\tau \in \mathbb{C}$, $\text{Im}\tau > 0$, and $\theta(z)$ the theta-function $\theta(z) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z)$. The Baker–Akhiezer function has the form

$$\psi(n, z) = \frac{\theta(z + c + nh)}{\theta(z)} \left(\frac{\theta(z - h)}{\theta(z)} \right)^n, \quad c, h \notin \{\mathbb{Z} + \tau\mathbb{Z}\}. \quad (15)$$

For the meromorphic function $\lambda = \frac{\theta(z-a_1)\dots(z-a_k)}{\theta^k(z)}$, $a_1 + \dots + a_k = 0$ there is a unique operator $L(\lambda) = v_k(n)T^k + \dots + v_0(n)$ such that $L(\lambda)\psi = \lambda\psi$. Coefficients of $L(\lambda)$ can be found from the last identity (see [39]). Operators $L(\lambda)$ for different λ form a commutative ring of difference operators. This example can be generalized from the elliptic spectral curves to the principle polarized abelian spectral varieties. It allows to construct commuting difference operators in several discrete variables with matrix coefficients (see [39]).

At $l > 1$ common eigenfunctions cannot be found explicitly. This is the main difficulty for constructing higher rank operators and higher rank solutions of the 2D-Toda chain. Recall the needed results of [2]. One-point commuting operators of rank l have the form

$$L = \sum_{i=-Nr_-}^{Nr_+} u_i(n)T^i, \quad A = \sum_{i=-Mr_-}^{Mr_+} v_i(n)T^i,$$

where $l = r_- + r_+$, $(N, M) = 1$. Consider the space $\mathcal{H}(z)$ of solutions of the equation $Ly = zy$. We have $\dim \mathcal{H}(z) = N(r_- + r_+)$. The operator A defines the linear operator $A(z)$ on $\mathcal{H}(z)$. Let us choose the basis $\varphi^i(n)$ in $\mathcal{H}(z)$, satisfying the normalization conditions $\varphi^i(n) = \delta_{in}$, $-Nr_- \leq i, n < Nr_+$. The components of $A(z)$ in the basis $\varphi^i(n)$ are polynomials in z . The characteristic polynomial of $A(z)$ has the form $\det(w - A(z)) = R^l(w, z)$. Polynomial R defines the spectral curve Γ , i.e.

$$L\psi = z\psi, \quad A\psi = w\psi, \quad R(z, w) = 0.$$

Common eigenfunctions of L and A form a vector bundle of rank l over the affine part of Γ . Let us choose the basis in the space of common eigenfunctions such that

$$\psi_n^i(P) = \delta_{in}, \quad -r_- \leq i, n < r_+, \quad P = (z, w) \in \Gamma.$$

Functions $\psi_n^i(P)$ have the pole divisor $\gamma = \gamma_1 + \dots + \gamma_{lg}$ of degree lg . We have the following identities

$$\alpha_s^j \text{Res}_{\gamma_s} \psi_n^i(P) = \alpha_s^i \text{Res}_{\gamma_s} \psi_n^j(P).$$

The pair (γ, α) is called the *Tyurin parameters*, where α is the set of vectors

$$\alpha_1, \dots, \alpha_{lg}, \quad \alpha_s = (\alpha_s^{-r_-}, \dots, \alpha_s^{r_+-1}).$$

The Tyurin parameters define a stable holomorphic vector bundle on Γ of degree lg with holomorphic sections $\zeta_{-r_-}, \dots, \zeta_{r_+-1}$, where γ is the divisor of their linear dependence $\sum_{j=-r_-}^{r_+-1} \alpha_s^j \zeta_j(\gamma_s) = 0$. Let $\Psi(n, P)$ be the Wronski matrix with the components $\Psi^{ij}(n, P) = \psi_{n+j}^i(P)$, $-r_- \leq i, j < r_+$. Function $\det \Psi(n, P)$ is holomorphic in the neighbourhood of $q = \infty$. The pole divisor of $\det \Psi(n, P)$ is γ , the zero divisor of $\det \Psi(n, P)$ is $\gamma(n) = \gamma_1(n) + \dots + \gamma_{lg}(n)$, herewith $\gamma(0) = \gamma$. Consider the matrix function $\chi(n, P) = \Psi(n+1, P)\Psi^{-1}(n, P)$,

$$\chi(n, P) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ \chi_{-r_-}(n, P) & \chi_{-r_-+1}(n, P) & \chi_{-r_-+2}(n, P) & \dots & \chi_{r_+-1}(n, P) \end{pmatrix}.$$

In the neighbourhood of q we have $\chi_i(n, k) = k^{-1}\delta_{i,0} - f_i(n, k)$, where k is a local parameter near q , $f_i(n, k)$ is an analytical function in the neighbourhood of q .

Theorem (I.M. Krichever, S.P. Novikov)

The matrix function $\chi(n, P)$ has simple poles in $\gamma_s(n)$, and

$$\alpha_s^j(n) \text{Res}_{\gamma_s(n)} \chi_i(n, P) = \alpha_s^i(n) \text{Res}_{\gamma_s(n)} \chi_j(n, P). \quad (16)$$

Points $\gamma_s(n+1)$ are zeros of $\det \chi(n, P)$

$$\det \chi(n, \gamma_s(n+1)) = 0. \quad (17)$$

Vectors $\alpha_j(n+1) = (\alpha_s^{-r-}(n+1), \dots, \alpha_s^{r+-1}(n+1))$ satisfy

$$\alpha_s(n+1) \chi(n, \gamma_s(n+1)) = 0. \quad (18)$$

Equations (16)–(18) define the discrete dynamics of the Tyurin parameters. In [2] solutions of (16)–(18) are found at $g = 1, l = 2$. The corresponding operators in the simplest case have the form

$$L = L_2^2 - \wp(\gamma_n) - \wp(\gamma_{n-1}),$$

L_2 is the difference Schrödinger operator $L_2 = T + v_n + c_n T^{-1}$ with the coefficients

$$c_n = \frac{1}{4}(s_{n-1}^2 - 1)F(\gamma_n, \gamma_{n-1})F(\gamma_{n-2}, \gamma_{n-1}), \quad v_n = \frac{1}{2}(s_{n-1}F(\gamma_n, \gamma_{n-1}) - s_n F(\gamma_{n-1}, \gamma_n)),$$

where

$$F(u, v) = \zeta(u+v) - \zeta(u-v) - 2\zeta(v).$$

Here, $\wp(u)$, $\zeta(u)$ are the Weierstrass functions, s_n , γ_n are the functional parameters.

3 Proof of Theorems 1–4

Let Γ be the hyperelliptic spectral curve (1), L_4 , L_{4g+2} are operators of the form (2) with the properties (3). Matrix $\chi(n, P) = \Psi(n+1, P)\Psi^{-1}(n, P)$ has the form

$$\chi(n, P) = \begin{pmatrix} 0 & 1 \\ \chi_1(n, P) & \chi_2(n, P) \end{pmatrix}.$$

The functions χ_1 , χ_2 have the following expansions in the neighbourhood of $q = \infty$:

$$\chi_1(n) = b_0(n) + b_1(n)k + \dots, \quad \chi_2(n) = 1/k + e_0(n) + e_1(n)k + \dots, \quad (19)$$

where $k = \frac{1}{\sqrt{z}}$.

Lemma 1 *The operator*

$$L_4 = T^2 + u_1(n)T + u_0(n) + u_{-1}(n)T^{-1} + u_{-2}(n)T^{-2}$$

has the coefficients:

$$u_1(n) = -e_0(n) - e_0(n+1), \quad u_0(n) = e_0^2(n) + e_1(n) - e_1(n+1) - b_0(n) - b_0(n+1),$$

$$u_{-1}(n) = b_0(n) \left(e_0(n) + e_0(n-1) - \frac{b_1(n-1)}{b_0(n-1)} \right) - b_1(n), \quad u_{-2}(n) = b_0(n)b_0(n-1).$$

If $b_1(n) = 0$, $e_0(n) = 0$, then L_4 can be written in the form (6), where

$$V_n = -b_0(n), \quad W_n = -e_1(n) - e_1(n+1).$$

Proof Using (4) let us express $\psi_{n+2}(P)$ and $\psi_{n-2}(P)$ via $\psi_{n-1}(P)$, $\psi_n(P)$, $\chi_1(n, P)$, $\chi_2(n, P)$

$$\psi_{n+2} = \psi_{n-1}\chi_1(n)\chi_2(n+1) + \psi_n(\chi_1(n+1) + \chi_2(n)\chi_2(n+1)), \quad \psi_{n-2} = \frac{\psi_n - \psi_{n-1}\chi_2(n-1)}{\chi_1(n-1)},$$

and substitute it in $L_4\psi_n = z\psi_n$. We get $P_1(n, P)\psi_n(P) + P_2(n, P)\psi_{n-1}(P) = z\psi_n(P)$, where

$$P_1(n) = \chi_1(n+1) + \chi_2(n+1)\chi_2(n) + u_1(n)\chi_2(n) + u_0(n) + \frac{u_{-2}(n)}{\chi_1(n-1)},$$

$$P_2(n) = \chi_2(n+1)\chi_1(n) + u_1(n)\chi_1(n) + u_{-1}(n) - u_{-2}(n)\frac{\chi_2(n-1)}{\chi_1(n-1)}.$$

Consequently we have

$$P_1 = z = \frac{1}{k^2}, \quad P_2 = 0. \tag{20}$$

From (19), (20) it follows that

$$\begin{aligned} P_1 - \frac{1}{k^2} &= \frac{e_0(n) + e_0(n+1) + u_1(n)}{k} + (b_0(n+1) + e_0(n)e_0(n+1) + e_1(n) + \\ &\quad e_1(n+1) + u_0(n) + e_0(n)u_1(n) + \frac{u_{-2}(n)}{b_0(n-1)}) + O(k) = 0, \\ P_2 &= \frac{b_0(n) - \frac{u_{-2}(n)}{b_0(n-1)}}{k} + (b_1(n) + b_0(n)e_0(n+1) + b_0(n)u_1(n) + \\ &\quad u_{-1}(n) + \frac{b_1(n-1)u_{-2}(n)}{b_0^2(n-1)} - \frac{e_0(n-1)u_{-2}(n)}{b_0(n-1)}) + O(k) = 0. \end{aligned}$$

This yields the formulas for the coefficients of L_4 . By direct calculations one can check that if $b_1(n) = e_0(n) = 0$, then L_4 has the form (6). Lemma 1 is proved.

Thus if χ_1, χ_2 satisfy (5), then $b_1(n) = e_0(n) = 0$, and hence L_4 has the form (6). Operators $L_4 - z$ and $L_{4g+2} - w$ have the common right divisor $T - \chi_2(n) - \chi_1(n)T^{-1}$, i.e.

$$L_4 - z = l_1(T - \chi_2(n) - \chi_1(n)T^{-1}), \quad L_{4g+2} - z = l_2(T - \chi_2(n) - \chi_1(n)T^{-1}),$$

where l_1 and l_2 are operators of orders 2 and $4g$. Let us assume that (5) holds. Then

$$(T + V_n T^{-1})^2 + W_n - z = (T + \chi_2(n+1) - \frac{V_{n-1}V_n}{\chi_1(n-1)}T^{-1})(T - \chi_2(n) - \chi_1(n)T^{-1}),$$

where χ_1, χ_2 satisfy the equations

$$V_{n-1}V_n + \chi_1(n-1)(V_n + V_{n+1} - z + W_n + \chi_1(n+1) + \chi_2(n)\chi_2(n+1)) = 0, \quad (21)$$

$$-V_{n-1}V_n\chi_2(n-1) + \chi_1(n-1)\chi_1(n)\chi_2(n+1) = 0. \quad (22)$$

We have $\det \chi(n, P) = -\chi_1(n, P) = \det \Psi(n+1, P)(\det \Psi(n, P))^{-1}$. The degree of the zero divisor $\gamma(n)$ of $\det \Psi(n, P)$ is $2g$. Since χ_1 is invariant under the involution σ , the divisor $\gamma(n)$ has the form

$$\gamma(n) = \gamma_1(n) + \sigma\gamma_1(n) + \dots + \gamma_g(n) + \sigma\gamma_g(n).$$

Let $\gamma_i(n)$ have the coordinates $(\mu_i(n), w_i(n))$. We introduce the polynomial in z

$$Q_n = (z - \mu_1(n)) \dots (z - \mu_g(n)).$$

From (17) we have $\chi_1(n, P) = b_0(n)\frac{Q_{n+1}}{Q_n}$, where $b_0(n)$ is some function. In the neighbourhood of q we have

$$\chi_1 = b_0(n) + b_2(n)k^2 + O(k^4).$$

By Lemma 1 $V_n = -b_0(n)$, so we get $\chi_1(n, P) = -V_n\frac{Q_{n+1}}{Q_n}$.

Since the pole divisor of $\chi_2(n, P)$ is $\gamma(n)$ and in the neighborhood of q we have (19), then $\chi_2(n, P) = \frac{w}{Q_n}$.

If $\chi_1(n, P) = -V_n\frac{Q_{n+1}}{Q_n}$ and $\chi_2(n, P) = \frac{w}{Q_n}$, then (22) holds identically, and (21) is reduced to (8). Theorem 1 is proved.

To prove Theorems 2–4 it is sufficient to prove that for potentials V_n, W_n from Theorems 2–4 there are polynomials $Q_n(z)$ of degree g in z which satisfy (9) (and hence satisfy (8)).

3.1 Theorem 2

Let $V_n = r_3n^3 + r_2n^2 + r_1n + r_0$, $W_n = g(g+1)r_3n$, then (9) takes the form

$$\begin{aligned} Q_{n-1}(n^3r_3 + n^2r_2 + nr_1 + r_0) + Q_n(z - 2n^3r_3 - n^2(2r_2 + 3r_3) - n(2r_1 + 2r_2 + 3r_3 + g(g+1)r_3) \\ - (2r_0 + r_1 + r_2 + r_3)) - Q_{n+2}(z - 2n^3r_3 - n^2(2r_2 + 9r_3) - \end{aligned}$$

$$n(2r_1 + 6r_2 + 15r_3 + g(1 + g)r_3) - (2r_0 + 3r_1 + 5r_2 + 9r_3 + g(g + 1)r_3) - Q_{n+3}(n^3r_3 + n^2(r_2 + 6r_3) + n(r_1 + 4r_2 + 12r_3) + r_0 + 2r_1 + 4r_2 + 8r_3) = 0. \quad (23)$$

Let us take the following ansatz for $Q_n(z)$

$$Q_n = \delta_g n^g + \dots + \delta_1 n + \delta_0, \quad \delta_i = \delta_i(z),$$

then (23) can be rewritten in the form

$$\beta_{g+3}(z)n^{g+3} + \beta_{g+2}(z)n^{g+2} + \dots + \beta_0(z) = 0$$

for some $\beta_s(z)$. Potentials V_n, W_n have the following remarkable properties: it turns out that

$$\beta_g = \beta_{g+1} = \beta_{g+2} = \beta_{g+3} = 0$$

automatically (this can be checked by direct calculations). From (23) we find β_s

$$\begin{aligned} \beta_s = & r_3(2s + 1)(g(g + 1) - s(s + 1))\delta_s + \sum_{m=1}^g ((-1)^m (C_{s+m}^m r_0 - C_{s+m}^{m+1} r_1 + \\ & C_{s+m}^{m+2} r_2 - C_{s+m}^{m+3} r_3) + 2^m (C_{s+m}^m (2r_0 + 3r_1 + 5r_2 + 9r_3 + g(g + 1)r_3 - z) + \\ & 2C_{s+m}^{m+1} (2r_1 + 6r_2 + 15r_3 + g(g + 1)r_3) + 4C_{s+m}^{m+2} (2r_2 + 9r_3) + 16C_{s+m}^{m+3} r_3) - \\ & 3^m (C_{s+m}^m (r_0 + 2r_1 + 4r_2 + 8r_3) + 3C_{s+m}^{m+1} (r_1 + 4r_2 + 12r_3) + \\ & 3^m (C_{s+m}^m (r_0 + 2r_1 + 4r_2 + 8r_3) + 3C_{s+m}^{m+1} (r_1 + 4r_2 + 12r_3) + \\ & 9C_{s+m}^{m+2} (r_2 + 6r_3) + 27C_{s+m}^{m+3} r_3)) \delta_{s+m}, \end{aligned} \quad (24)$$

where $0 \leq s < g - 1$, $C_m^k = \frac{m!}{k!(m-k)!}$ at $m \geq k$, $C_m^k = 0$ at $m < k$, δ_g is a constant and $\delta_s = 0$, if $s > g$. From $\beta_s = 0$ we express δ_s via $\delta_{s+1}, \dots, \delta_g$. In particular,

$$\delta_{g-1} = \frac{\delta_g(2g^2r_2 + g(g + 1)r_3 + 2z)}{2(2g - 1)r_3}.$$

For a suitable δ_g we have $Q_n = z^g + \alpha_{g-1}(n)z^{g-1} + \dots + \alpha_0(n)$. So we proved that there exists Q_n satisfying (9). Theorem 2 is proved.

In [5] it was proved that

$$\mathcal{L}_4^\sharp = (\partial_x^2 + r_3x^3 + r_2x^2 + r_1x + r_0)^2 + g(g + 1)r_3x$$

commutes with a differential operator $\mathcal{L}_{4g+2}^\sharp$ of order $4g + 2$. The operator L_4^\sharp is a discrete analogue of \mathcal{L}_4^\sharp . At $g = 1$ the operators $\mathcal{L}_4^\sharp, \mathcal{L}_6^\sharp$ were found by Dixmier [10]. The operators $\mathcal{L}_4^\sharp, \mathcal{L}_{4g+2}^\sharp$ define a commutative subalgebra in the first Weyl algebra.

Let us consider the algebra W generated by two elements p and q with the relation $[p, q] = p$. Since $[T, n] = T$, the algebra is isomorphic to the algebra of difference operators with polynomial coefficients. The algebra W has the following automorphisms

$$H : W \rightarrow W, \quad H(p) = p, \quad H(q) = q + G(p),$$

where G is an arbitrary polynomial. Operators $L_4^\sharp, L_{4g+2}^\sharp$ define the commutative subalgebra in W . Consequently, if we replace $n \rightarrow n + G(T)$ in $L_4^\sharp, L_{4g+2}^\sharp$, then we obtain the new commuting difference operators with polynomial coefficients.

3.2 Theorem 3

Let $V_n = r_1 a^n + r_0$, $W_n = (a^{2g+1} - a^{g+1} - a^g + 1)r_1 a^{n-g}$, then (9) takes the form

$$\begin{aligned} & Q_{n-1}(r_0 + a^n r_1) + Q_n(z - 2r_0 - a^{n-g} r_1 - a^{n+g+1} r_1) + \\ & Q_{n+2}(2r_0 + a^{n+1-g} r_1 + a^{n+g+2} r_1 - z) - Q_{n+3}(r_0 + a^{n+2} r_1) = 0. \end{aligned} \quad (25)$$

Let $Q_n = B_g a^{gn} + B_{g-1} a^{(g-1)n} + \dots + B_1 a^n + B_0$, $B_i = B_i(z)$. We introduce the notation $y = a^n$, then $Q_n = B_g y^g + \dots + B_0$, and (25) takes the form

$$\begin{aligned} & \sum_{s=0}^g B_s (a^{-g-s} (a^g - a^s) (a^{g+s+1} - 1) (a^{2s+1} - 1) r_1 y^{s+1} - a^{-s} (a^{2s} - 1) ((a^s - 1)^2 r_0 + a^s z) y^s) = \\ & \sum_{s=1}^g y^s (B_s a^{-s} (1 - a^{2s}) ((a^s - 1)^2 r_0 + a^s z) + B_{s-1} a^{1-g-s} (a^g - a^{s-1}) (a^{g+s} - 1) (a^{2s-1} - 1) r_1) = 0. \end{aligned}$$

Hence we obtain

$$B_{s-1} = B_s \frac{a^{-s} (a^{2s} - 1) ((a^s - 1)^2 r_0 + a^s z)}{a^{1-g-s} (a^g - a^{s-1}) (a^{g+s} - 1) (a^{2s-1} - 1) r_1}, \quad s = 1, \dots, g.$$

Thus we found the polynomial Q_n , satisfying (9). Theorem 3 is proved.

The operator L_4^\vee is a discrete analogue of

$$\mathcal{L}_4^\vee = (\partial_x^2 + r_1 a^x + r_0)^2 + g(g+1) r_1 a^x$$

from [26], which commutes with a differential operator of order $4g+2$.

3.3 Theorem 4

Let $V_n = r_1 \cos(n) + r_0$, $W_n = -4r_1 \sin(\frac{g}{2}) \sin(\frac{g+1}{2}) \cos(n + \frac{1}{2})$. Equation (9) takes the form

$$\begin{aligned} & Q_{n-1}(r_0 + r_1 \cos(n)) + Q_n(z - 2r_0 - 2r_1 \cos(g + \frac{1}{2}) \cos(n + \frac{1}{2})) - \\ & Q_{n+2}(z - 2r_0 - 2r_1 \cos(g + \frac{1}{2}) \cos(n + \frac{3}{2})) - Q_{n+3}(r_0 + r_1 \cos(n + 2)) = 0. \end{aligned} \quad (26)$$

Let us take the following ansatz

$$Q_n = A_g \cos(gn) + A_{g-1} \cos((g-1)n) + \dots + A_1 \cos(n) + A_0, \quad A_i = A_i(z).$$

We substitute Q_n in (26) and after some simplifications we obtain

$$\begin{aligned} A_0 &= A_1 \frac{(z - 2r_0 + 2r_0 \cos(1)) \sin(1)}{2r_1 (\cos(g + \frac{1}{2}) - \cos(\frac{1}{2})) \sin(\frac{1}{2})}, \\ A_{s-1} &= \frac{A_s (z - 2r_0 + 2r_0 \cos(s)) \sin(s) + A_{s+1} r_1 (\cos(s - \frac{3}{2}) - \cos(g + \frac{1}{2})) \sin(s - \frac{3}{2})}{r_1 (\cos(g + \frac{1}{2}) - \cos(s - \frac{1}{2})) \sin(s - \frac{1}{2})}, \end{aligned}$$

where $2 \leq s \leq g$, $A_{g+1} = 0$, A_g is a suitable constant. We found Q_n satisfying (9). Theorem 4 is proved.

The operator L_4^\sharp is a discrete analogue of

$$\mathcal{L}_4^\sharp = (\partial_x^2 + r_1 \cos(x) + r_0)^2 + g(g+1)r_1 \cos(x)$$

from [29], which commutes with a differential operator of order $4g+2$.

Let us consider several examples.

Example 1 We introduce the notation $f(n) = r_3 n^3 + r_2 n^2 + r_1 n + r_0$. The operator

$$L_4^\sharp = (T + f(n)T^{-1})^2 + 2r_3 n$$

commutes with

$$\begin{aligned} L_6^\sharp = & T^3 + 3(f(n) + f'(n) + f''(n) + 4r_3)T + 3(f(n) + 3r_3 n + r_2)T^{-1} + \\ & (f(n-2)f'(n) + 2f''(n) - 8r_3)(f(n) - f'(n) + 3r_3 n - r_3 + r_2)f(n)T^{-3}. \end{aligned}$$

The spectral curve is

$$w^2 = z^3 + (2r_2 + 3r_3)z^2 + (r_1 r_3 + (r_2 + r_3)(r_2 + 3r_3))z + r_3((r_2 + r_3)(r_1 + r_2 + r_3) - r_0 r_3).$$

Example 2 The operator

$$L_4^\vee = (T + (r_1 a^n + r_0)T^{-1})^2 + r_1(a^3 - a^2 - a + 1)a^{n-1}$$

commutes with

$$\begin{aligned} L_6^\vee = & T^3 + (r_0(a+1+a^{-1}) + r_1 a^{n-1}(a^4 + a^2 + 1))T + (a+1+a^{-1})(r_1 a^n + r_0) \times \\ & (r_1 a^{n+1} - r_1 a^n + r_1 a^{n-1} + r_0)T^{-1} + (r_1 a^n + r_0)(r_1 a^{n-1} + r_0)(r_1 a^{n-2} + r_0)T^{-3}. \end{aligned}$$

The spectral curve is $w^2 = z^3 + \frac{2r_0(a-1)^2}{a}z^2 + \frac{r_0^2(a-1)^4}{a^2}z$.

Example 3 The operator

$$L_4^\sharp = (T + (r_1 \cos(n) + r_0)T^{-1})^2 - 4r_1 \sin(\frac{1}{2}) \sin(1) \cos(n + \frac{1}{2})$$

commutes with

$$\begin{aligned} L_6^\sharp = & T^3 + (2 \cos(1) + 1)(r_1(2 \cos(1) - 1) \cos(n+1) + r_0)T + \\ & (2 \cos(1) + 1)(r_1 \cos(n) + r_0)(r_1(2 \cos(1) - 1) \cos(n) + r_0)T^{-1} + \\ & (r_1 \cos(n-1) + r_0)(r_1 \cos(n-2) + r_0)(r_1 \cos(n) + r_0)T^{-3}. \end{aligned}$$

The spectral curve is $w^2 = z^3 - 8r_0 \sin^2(\frac{1}{2})z^2 - 8(r_1^2(\cos(1) + 1) - 2r_0^2) \sin^4(\frac{1}{2})z$.

Remark We see that the pairs of commuting operators L_4, L_{4g+2} and $\mathcal{L}_4, \mathcal{L}_{4g+2}$ have similar properties, and there are similar examples of such operators. It would be interesting to explain this duality. So far, our attempts to do so by some discretization were not successful.

Acknowledgements The authors are supported by a Grant of the Russian Federation for the State Support of Researches (Agreement No 14.B25.31.0029).

Appendix

Consider the differential–difference system (13), (14).

Theorem 5 *Let us assume that the potentials $V_n(t)$, $W_n(t)$ of $L_4 = (T + V_n(t)T^{-1})^2 + W_n(t)$ satisfy (13), (14). We additionally assume that $[L_4, L_{4g+2}] = 0$ for some difference operator L_{4g+2} . Then $Q_n(t)$ associated with L_4 satisfies the evolution equation*

$$\dot{Q}_n = V_n(Q_{n+1} - Q_{n-1}). \quad (27)$$

Equation (27) defines the symmetry of (8). At $g = 1$ functions $V_n(t)$, $W_n(t)$ can be expressed via $\gamma_n(t)$ using (9). In this case the system (13), (14) and the equation (27) are reduced to one equation

$$\dot{\gamma}_n = \frac{F_1(\gamma_n)(\gamma_{n-1} - \gamma_{n+1})}{(\gamma_{n-1} - \gamma_n)(\gamma_n - \gamma_{n+1})}. \quad (28)$$

This equation is a discrete analogue of the Krichever–Novikov equation, which appeared in the theory of rank two solutions of KP [1]. Equations similar to (13), (14) and (28) were considered in [40], [41].

Proof Using $(\partial_t - V_{n-1}(t)V_n(t)T^{-2})\psi_n = 0$, and (4) let us express $\dot{\psi}_{n-1}$, $\dot{\psi}_n$, $\dot{\psi}_{n+1}$, ψ_{n-2} , ψ_{n-3} in terms of ψ_{n-1} , ψ_n , $\chi_1(n)$, $\chi_2(n)$

$$\begin{aligned} \dot{\psi}_{n-1} &= V_{n-2}V_{n-1}\psi_{n-3}, & \dot{\psi}_n &= V_{n-1}V_n\psi_{n-2}, & \dot{\psi}_{n+1} &= V_nV_{n+1}\psi_{n-1}, \\ \psi_{n-2} &= \frac{\psi_n - \psi_{n-1}\chi_2(n-1)}{\chi_1(n-1)}, \\ \psi_{n-3} &= \frac{\psi_{n-1}(\chi_1(n-1) + \chi_2(n-2)\chi_1(n-1)) - \psi_n\chi_2(n-2)}{\chi_1(n-2)\chi_1(n-1)}. \end{aligned}$$

From (4) it follows that

$$\dot{\psi}_{n+1} - \chi_1(n)\dot{\psi}_{n-1} - \dot{\chi}_1(n)\psi_{n-1} - \chi_2(n)\dot{\psi}_n - \dot{\chi}_2(n)\psi_n = \mathcal{A}_n\psi_n + \mathcal{B}_n\psi_{n-1} = 0,$$

where

$$\begin{aligned} \mathcal{A}_n &= V_{n-2}V_{n-1}\chi_1(n)\chi_2(n-2) - \chi_1(n-2)(V_{n-1}V_n\chi_2(n) + \chi_1(n-1)\dot{\chi}_2(n)), \\ \mathcal{B}_n &= V_{n-2}V_{n-1}\chi_1(n)(\chi_1(n-1) + \chi_2(n-2)\chi_2(n-1)) + \\ &+ V_n\chi_1(n-2)(V_{n+1}\chi_1(n-1) + V_{n-1}\chi_2(n-1)\chi_2(n)) - \chi_1(n-2)\chi_1(n-1)\dot{\chi}_1(n). \end{aligned}$$

Consequently, we have $\mathcal{A}_n = \mathcal{B}_n = 0$. For $\chi_1 = -V_n(t)\frac{Q_{n+1}(t)}{Q_n(t)}$, $\chi_2 = \frac{w}{Q_n(t)}$ it follows from (8) and from $\mathcal{A}_n = \mathcal{B}_n = 0$ that $Q_n(t)$ satisfies (27). Theorem 5 is proved.

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